Calculus

Lau Chi Hin

The Chinese University of Hong Kong

Calculus

- Its History
- Its Importance
- Its Development

Calculus

Calculus = Differentiation + Integration

Zeno's paradox

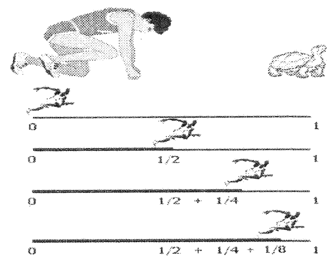


Figure 1: *The Runner's Paradox*

Contributors

Archimedes	Newton
Fermat	Leibniz
Descartes	Cauchy
Barrow	Weierstrass

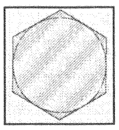
Archimedes (BC.287-BC.212)

Archimedes used geometric and mechanical method to find

1. approximate value of π
2. volume of a sphere
3. area under a parabola

Archimedes (BC.287-BC.212)

By considering inscribed and circumscribed regular polygons, he obtained



$$3.1408 < \pi < 3.1429$$

Archimedes (BC.287-BC.212)

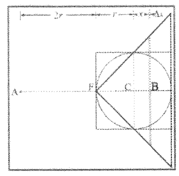
moment of slices of sphere and cone placed at C

$$= (\pi(r^2 - x^2) + \pi(r + x)^2)\Delta x \times r$$

$$= 2\pi(r^2 + rx)r\Delta x$$

$$= 2 \times \pi r^2 \Delta x \times (r + x)$$

= *moment of 2 slices of cylinder placed at B*



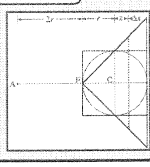
Archimedes (BC.287-BC.212)

sphere + cone concentrated at C | balance with | 2 cylinders

$$\left(V + \frac{1}{3}\pi(2r)^2(2r)\right) \times r = 2 \times \pi r^2 (2r) \times r$$

$$V = 4\pi r^3 - \frac{8}{3}\pi r^3$$

$$V = \frac{4}{3}\pi r^3$$



Archimedes (BC.287-BC.212)

Area under $y = x^2$

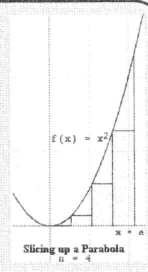
$$= \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n}$$

$$= \lim_{n \to \infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

$$= \lim_{n \to \infty} \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{1}{3}$$

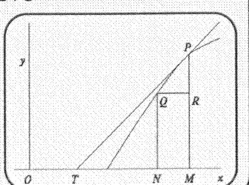


Differential Calculus

tangent line of a curve

$$TM : MP \approx QR : RP$$

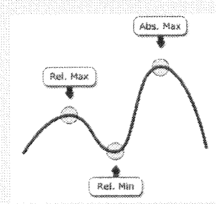
$$TM = \frac{f(x_o)e}{f(x_o) - f(x_o - e)}$$



Differential Calculus

Maximum or Minimum value

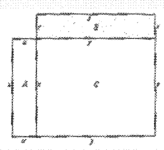
$$f(x + e) \approx f(x)$$



Differential Calculus

When is the area of a rectangle with fixed perimeter maximum?

$$(a+e)(b-e) \approx ab$$
$$ab + be - ae - e^2 \approx ab$$
$$(b-a)e - e^2 \approx 0$$
$$b - a = 0$$

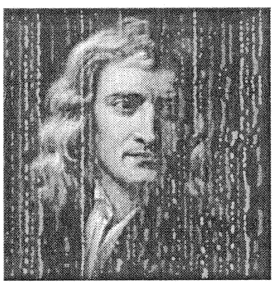


Area is maximum when $a=b$

Birth of Calculus

Integration	inverse ⟺	Differentiation
Integral	=	Anti-derivative

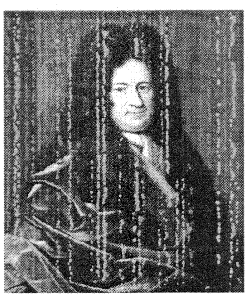
Isaac Newton (1643-1727)



Isaac Newton (1643-1727)

- View calculus as a tool to study physics and geometry.
- Extend Binomial Theorem to fractional and negative exponents.
- Derive the 3 Kepler's law of planetary motion assuming the inverse square law.
- Propose Law of Universal Gravitation.
- Publish "Mathematical principles of natural philosophy" (1687).

Gottfried Leibniz (1646-1716)



Gottfried Leibniz (1646-1716)

- Believe that calculus was a metaphysical explanation of change.
- Study tangent problem.
- Systematic study of rules of differentiation.
- Publish "New method for maximums and minimums".
- Use of the symbols $\int$ and dx.

Fundamental theorem of Calculus

First part:

Let *f(x)* be a continuous function defined on *[a,b]*, then

$$F(x) = \int_a^x f(t)\,dt$$

F(x) is a continuous on *[a,b]*, differentiable on (a,b) and

$$F'(x) = f(x)$$

Fundamental theorem of Calculus

Second part:

Let *f(x)* be a continuous function defined on *[a,b]*, and *F(x)* is a function such that

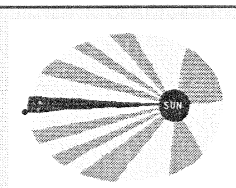
$$F'(x) = f(x)$$

Then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Kepler's Law of planetary motion

1. The orbit is an ellipse with the sun at one of the foci.
2. A line joining a planet and the sun sweeps out equal areas in equal time.
3. The squares of the orbital periods are directly proportional to the cubes of the semi-major axes.



Kepler's Law of planetary motion

Inverse square law	$\Rightarrow$	$T^2 \propto R^3$
Conservation of momentum	$\Rightarrow$	Equal area
Differential equation	$\Rightarrow$	Elliptic orbit

Kepler's Law of planetary motion

Centripetal force:

$$F = \frac{mv^2}{R} = \frac{m}{R}\left(\frac{2\pi R}{T}\right)^2 \propto \frac{R}{T^2}$$

Assume inverse square law

$$F \propto \frac{1}{R^2}$$

Then

$$\frac{1}{R^2} \propto \frac{R}{T^2}$$

$$T^2 \propto R^3$$

Kepler's Law of planetary motion

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

is constant

$\Rightarrow$

Angular momentum

$$L = m\bar{r} \times \bar{v} = mr^2\dot{\theta}$$

is constant

Kepler's Law of planetary motion

Newton second Law: $\frac{\vec{F}}{m} = \vec{a}$

$$-\frac{GM}{r^2}\hat{e}_r = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$\Rightarrow \begin{cases} \ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \end{cases}$$

Kepler's Law of planetary motion

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$
$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0$$
$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$
$$r^2\dot{\theta} = l$$
$$\dot{\theta} = \frac{l}{r^2}$$

In fact, this is known already from conservation of angular momentum.

Kepler's Law of planetary motion

$$-\frac{GM}{r^2} = \ddot{r} - r\dot{\theta}^2$$
$$\ddot{r} - r\left(\frac{l}{r^2}\right)^2 = -\frac{GM}{r^2}$$

Therefore we need to solve

$$\ddot{r} - \frac{l^2}{r^3} = -\frac{GM}{r^2}$$

Kepler's Law of planetary motion

Let $a = \frac{l^2}{GM}$ and $u = \frac{a}{r}$

$$\dot{r} = \frac{d}{dt}\left(\frac{a}{u}\right) = \frac{d\theta}{dt}\frac{d}{d\theta}\left(\frac{a}{u}\right) = -\frac{lu^2}{a^2}\cdot\frac{a}{u^2}u' = \frac{lu'}{a}$$
$$\ddot{r} = -\frac{d}{dt}\left(\frac{lu'}{a}\right) = -\frac{l}{a}\frac{d\theta}{dt}\frac{d}{d\theta}u' = -\frac{l^2u^2u''}{a^3}$$

Kepler's Law of planetary motion

$$\ddot{r} - \frac{l^2}{r^3} = -\frac{GM}{r^2}$$
$$-\frac{l^2u^2u''}{a^3} - \frac{l^2u^3}{a^3} = -\frac{l^2u^2}{a^3}$$

The equation is simplified to

$$u'' + u = 1$$

Kepler's Law of planetary motion

The general solution is

$$u'' + u = 1$$
$$u = 1 + \varepsilon\cos(\theta - \alpha)$$

$$r = \frac{a}{1 + \varepsilon\cos(\theta - \alpha)}$$

Recall: $u = \frac{a}{r}$

which represents a conic curve with focus at the origin.

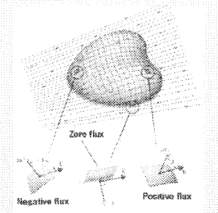
Edmond Halley (1656-1742)

- Claim that the comet sightings of 1456, 1531, 1607 and 1682 related to the same comet.
- Predicted that the comet would return in 1758.
- The Halley's comet was seen again on 25th Dec 1758.

Electromagnetism

Gauss' Law

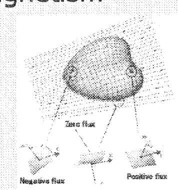
$$\oiint_S \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}$$



Electromagnetism

Gauss' Law for magnetism

$$\oiint_S \vec{B}\cdot d\vec{A} = 0$$



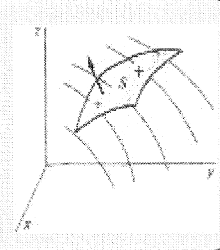
Electromagnetism

Faraday's Law

$$\oint_{\partial S} \vec{E}\times d\vec{l} = -\frac{\partial \Phi_B}{\partial t}$$

where

$$\Phi_B = \iint_S \vec{B}\cdot d\vec{A}$$



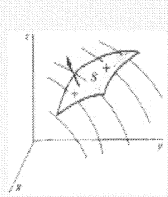
Electromagnetism

Ampere's Law

$$\oint_{\partial S} \vec{B}\times d\vec{l} = \mu_0 I + \mu_0\varepsilon_0\frac{\partial \Phi_E}{\partial t}$$

where

$$\Phi_E = \iint_S \vec{E}\cdot d\vec{A}$$



Maxwell's Equations

Name	Integral form	Differential form
Gauss' Law	$\oiint_S \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}$	$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$
Gauss' Law	$\oiint_S \vec{B}\cdot d\vec{A} = 0$	$\vec{\nabla}\cdot\vec{B} = 0$
Faraday's Law	$\oint_{\partial S} \vec{E}\times d\vec{l} = -\frac{\partial \Phi_B}{\partial t}$	$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Ampere's Law	$\oint_{\partial S} \vec{B}\times d\vec{l} = \mu_0 I + \mu_0\varepsilon_0\frac{\partial \Phi_E}{\partial t}$	$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$

Electromagnetic wave

In vacuum, Maxwell's equations become

$$\bar{\nabla} \cdot \bar{E} = 0$$
$$\bar{\nabla} \cdot \bar{B} = 0$$
$$\bar{\nabla} \times \bar{E} = -\frac{\partial B}{\partial t}$$
$$\bar{\nabla} \times \bar{B} = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

Electromagnetic wave

Using the identity $\bar{\nabla} \times (\bar{\nabla} \times \bar{A}) = \bar{\nabla}(\bar{\nabla} \cdot \bar{A}) - \bar{\nabla}^2 \bar{A}$

We have

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{E}) = \bar{\nabla}(\bar{\nabla} \cdot \bar{E}) - \bar{\nabla}^2 \bar{E}$$
$$\bar{\nabla} \times \left(-\frac{\partial B}{\partial t}\right) = -\bar{\nabla}^2 \bar{E}$$
$$-\frac{\partial}{\partial t}(\bar{\nabla} \times B) = -\bar{\nabla}^2 \bar{E}$$
$$-\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\right) = -\bar{\nabla}^2 \bar{E}$$
$$\bar{\nabla}^2 \bar{E} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

Electromagnetic wave

$$\bar{\nabla}^2 \bar{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2}$$

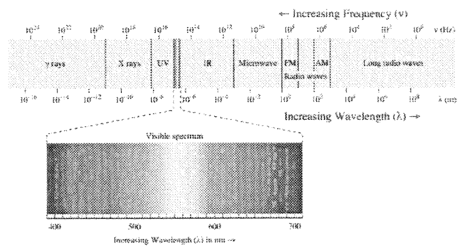
The above equation shows the existence of wave of oscillating electric and magnetic fields which travel at a speed

$$\frac{1}{\sqrt{\mu_0 \varepsilon_0}} \approx 300{,}000 \; kms^{-1}$$

which is very close to the speed of light.

Maxwell then claimed that light is in fact an electromagnetic wave.

Electromagnetic wave



Maxwell's equation in tensor form

$$\begin{cases} F^{\alpha\beta}{}_{,\alpha} = \frac{4\pi}{c} J^{\beta} \\ F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \end{cases}$$

where

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

is the electromagnetic tensor and

$$J^{\alpha} = (c\rho, J_x, J_y, J_z)$$

is the 4-current.

Nevier-Stokes Equation

Nevier-Stokes Equation describe the motion of viscous fluid.

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla \mathbf{p} + \mu \Delta \mathbf{v} + \mathbf{f}$$

where
- $\mathbf{v}$: velocity
- ρ : density
- p : pressure
- $\mathbf{f}$: external force

The continuity equation reads $\nabla \cdot \mathbf{v} = 0$

Einstein field equation

According to General Relativity, gravity is described as a curved space time caused by matter and energy.

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = -\frac{8\pi G}{c^4} T_{\alpha\beta}$$

$R_{\alpha\beta}$: Ricci tensor

R : scalar curvature

$g_{\alpha\beta}$: metric tensor

$T_{\alpha\beta}$: energy-momentum-stress tensor

Schrödinger equation

In quantum mechanics, particles are described by wave function satisfying

$$i \frac{h}{2\pi} \frac{d\psi}{dt} = H\psi$$

where

h : Planck's constant

ψ : wave function

H : Hamiltonian operator

Black-Scholes' equation

Black-Scholes model the price of an option by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where V : price of the option

S : price of the underlying instrument

σ : volatility

r : constant interest rate

Other useful tools

1. Power series (Taylor series)
2. Newton's method of approximation
3. Differential equations
4. Lagrange multipliers
5. Calculus of variation
6. Laplace/Fourier transform
7. Lebesgue integration

The End

Lau Chi Hin:
chlau@math.cuhk.edu.hk